SOME COEFFICIENT ESTIMATES FOR H^p FUNCTIONS

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Abstract. We find the maximum modulus of the n-th Taylor coefficient c_n of a function in the unit ball of H^p, $1 \le p \le \infty,$ provided that c_0 is fixed, and identify the corresponding extremal functions.

1. Introduction

and c such that $0 < c < 1.$ In the following sections, we consider only such values of p and c. In proving the main result, we prove some intermediate theorems which are of independent interest.

3. Statement of the Main Results

Theorem 3.1. *If $2^{-\frac{1}{p}} \le c \le 1$; then*

$$M_p(n;c) = \frac{2}{p}c^{1-\frac{p}{2}}\sqrt{1-c^p}$$

and the corresponding extremal function is

$$f(z) = (c^{\frac{p}{2}} + \sqrt{1-c^p}z^n)^{\frac{2}{p}}.$$

Theorem 3.2. *If $0 < c < 2^{-\frac{1}{p}}$; then the zero-free function f such that $\|f\|_p \le 1$ and $|f(0)| = c$ that maximizes $|f'(0)|$ is*

$$f(z) = 2^{-\frac{1}{p}}(1+z)^{\frac{2}{p}}(2^{\frac{1}{p}}c)^{\frac{1-z}{1+z}}$$

and

$$f'(0) = c\left(\frac{2}{p} + \log\frac{1}{2^{\frac{2}{p}}c^2}\right).$$

Theorem 3.3. *If $0 < c < 2^{-\frac{1}{p}}$; then*

$$M_p(n;c) = \left(\frac{2}{p} - 1\right)cv + \frac{c}{v}$$

and the corresponding extremal function is

$$f(z) = \frac{c}{v}(1+vz^n)^{\frac{2}{p}-1}(v+z^n)$$

where v is the unique root ($0 < v \le 1$) of $v^p - c^p = c^pv^2.$ In particular, for $p = 1$ and $0 < c < \frac{1}{2}$; $M_1(n;c) = 1$ and $f(z) = c + z^n + cz^{2n}.$

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Proposition 4.2. *The singular function $S(z)$ that maximizes $\mathrm{Re}\,S'(0)$ if $S(0) = c$ is*

$$S(z) = c^{1-}$$

We now have a variational problem, where we need to find

$$\max \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)\cos t\,dt$$

under the constraints $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\varphi(t)}dt = 1$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)dt = \log c < 0.$

If this maximum equals μ and is attained when $\varphi(t) = \varphi_0(t);$ then φ_0 also solves the following dual variational problem: find

$$\min \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\varphi(t)}dt$$

under the constraints $\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)\cos t\,dt = \mu$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)dt = \log c;$ because the above minimum is then equal to 1. To see this, suppose that for some $\varphi(t) = \varphi_1(t)$ satisying the constraints of the dual problem $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\varphi_1(t)}dt < 1.$ Then there is some $s > 0$ such that the function $\varphi_2(t) = \varphi_1(t) + s\cos t$ satisfies $\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_2(t)\cos t\,dt$ s >

Therefore,

$$\begin{aligned}
\varphi'(v) \geq 0 \quad &\Leftrightarrow \quad c^{-\frac{p}{2}} \frac{v^{p-1}}{\sqrt{v^p - c^p}} \geq 1 + \frac{1}{v^2} \\
&\Leftrightarrow \quad (1+v^2)^2 (c^p)^2 - v^p (1+v^2)^2 c^p + v^{2(p+1)} \geq 0 \\
&\Leftrightarrow \quad (c^p - \frac{v^p}{1+v^2})(c^p - \frac{v^{2+p}}{1+v^2}) \geq 0
\end{aligned}$$

When $c \geq 2^{-\frac{1}{p}}$, $\varphi'(v) \geq 0$ and the maximum of $\varphi(v)$ is obtained at $v = 1$:

$$\varphi(1) = \frac{2}{p} c^{1-\frac{p}{2}} \sqrt{1 - c^p}.$$

In that case, the function

$$f(z) = (c^{\frac{p}{2}} + \sqrt{1 - c^p}\, z)^{\frac{2}{p}}$$

is an element of H^p with norm 1 such that $f(0) = c$ and $f'(0) = \varphi(1)$.

When $n > 1$, use the function $\tilde{f}$ described in Section 2. Since $f(z) = \tilde{f}(z^n)$, we obtain the extremal function

$$f(z) = (c^{\frac{p}{2}} + \sqrt{1 - c^p}\, z^n)^{\frac{2}{p}}$$

with the same maximal n-th Taylor coefficient as in the case $n = 1$. ¤

Notice that f is a zero-free function, and therefore Theorem 3.1 also solves the extremal problem for zero-free H^p functions whose value at the origin is not too close to 0. Let us now consider zero-free functions in H^p whose value at the origin are small, as stated in Theorem 3.2.

Proof. Let $0 < c < 2^{-\frac{1}{p}}$ and let $f \in H^p$ be a non-zero function such that $f(0) = c$ and $\|f\|_p \leq 1$ for which $|f'(0)|$ is maximal. Write $f(z) = S(z)F(z)$ where S is a singular function and F is an outer function. Writing $S(0) = u$ and $F(0) = v$, notice that by Proposition 4.3, $v \geq 2^{-\frac{1}{p}}$. Using the estimates given by Proposition 4.2 and Theorem 3.1, we get that

$$\begin{aligned}
|f'(0)| &\leq v 2u \log \frac{1}{u} + u \frac{2}{p} v^{1-\frac{p}{2}} \sqrt{1 - v^p} \\
&= 2c \log \frac{v}{c} + \frac{2c}{p} \sqrt{\frac{1}{v^p} - 1} \\
&= \varphi(v)
\end{aligned}$$

One can easily show that $\varphi(v)$ is decreasing on $[2^{-\frac{1}{p}}, 1]$ and therefore attains its maximum at $v = 2^{-\frac{1}{p}}$. Therefore $u = c 2^{\frac{1}{p}}$, and the function

$$f(z) = (2^{\frac{1}{p}} c)^{\frac{1-z}{1+z}} 2^{-\frac{1}{p}} (1+z)^{\frac{2}{p}}$$

is a zero-free function such that $f(0) = c;\ \|f\|_p = 1$ and

$$f'(0) = \varphi(2^{-\frac{1}{p}}) = c\left(\frac{2}{p} + \log\frac{1}{2^{\frac{2}{p}}c^2}\right).$$

$\square$

We now consider functions in H^p that can have zeros and whose value at the origin is small, as stated in Theorem 3.3.

Proof. Consider the case $n = 1$ and let $f \in H^p$ be such that $\|f\|_p \le 1$ and $f(0) = c.$ Write $f(z) = B(z)F(z)$ where B is a Blaschke product with $B(0) = v > 0;$ and F is zero-free with $F(0) = u.$

Suppose first that $u \ge 2^{-\frac{1}{p}}.$ Then $c \le v \le 2^{\frac{1}{p}}c;$ so by the proof of Theorem 3.1

$$\begin{aligned} |f'(0)| &\le c\left(\frac{1}{v} - v\right) + \frac{2}{p}c^{1-\frac{p}{2}}\sqrt{v^p - c^p} \\ &= \varphi(v) \end{aligned}$$

and

$$\varphi'(v) \ge 0 \iff \left(c^p - \frac{v^p}{1+v^2}\right)\left(c^p - \frac{v^{2+p}}{1+v^2}\right) \ge 0$$

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