

JENSEN TYPE INEQUALITIES AND RADIAL NULL SETS

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Abstract. We extend Jensen's formula to obtain an upper estimate of $\log |f(0)|$ for analytic functions in the unit disk \mathbf{D} that are subject to a growth restriction. Suppose we have a closed subset E of the unit circle and f in addition is continuous in the union of the open disk and E . We obtain a formula that gives an upper estimate of $\log |f(0)|$ in terms of the values of f on E and the so-called k -entropy of E . When the set E is taken to be the whole unit circle, we get the classical Jensen's inequality. Our formula is then applied to the study of radial null sets. 2000 Mathematics Subject Classification: 30H05, 30E25, 46E15.

1 Growth Spaces

In what follows, k denotes an increasing twice differentiable function that maps $[0; 1) \rightarrow [0; \infty)$.

2 Two Problems

(A) Find good (upper and lower) estimates for the quantity

$$J(Z; k) = \sup \{ \log \int f(0) j : f \in UBA^{<k>}; \int j_Z = 0 \}$$

where $Z = \{a_n\} \subset \mathbb{D}$ is a given sequence.

(B) Find good estimates for

$$J(E; \nu; k) = \sup \{ \log \int f(0) j : f \in UBA^{<k>} \setminus C(\mathbb{D} \setminus E); \int j_E = \nu \}$$

where $E \subset \mathbb{D}$ is a closed set and ν is a non-negative continuous function on E :

Note that for $k \leq 0$; ($UBA^{<0>} = H^1$) both problems have exact solutions:

$$J(Z; 0) = \sum_n \log \frac{1}{|a_n|}$$

$$J(E; \nu; 0) = \int_E \log \nu(z) dm(z)$$

where dm is the normalized Lebesgue measure on \mathbb{D} : (Here, we assume k

and the radial projection of S :

$$PrS = \left\{ \frac{z}{|z|} : z \in S \right\}$$

Then we have

$$J(Z; \mathbb{R}) \leq \inf_{S \subset Z} \left[\int_S f^{\otimes}(\Pr S) + \log \int_S f^{\otimes}(\Pr S) \right] + \int_S T(s) + \mathbb{R} \log^+ T(s) g + C_{\mathbb{R}}$$

and

$$J(Z; \mathbb{R}) \geq \inf_{S \subset Z} \left[\int_S f^{\otimes}(\Pr S) + \log \int_S f^{\otimes}(\Pr S) \right] + \int_S T(s) g + C_{\mathbb{R}}$$

where $C_{\mathbb{R}} > 0$ depends only on \mathbb{R} ; and the in-ima are taken over all finite subsets S of Z :

COROLLARY 3.1 For a sequence Z such that 0 is not in Z ; define

$$D^+(Z) = \inf_{S \subset Z} \int_S f^{\otimes}(\Pr S) + \log \int_S f^{\otimes}(\Pr S) + \int_S T(s) g$$

Then $D^+(Z) < \mathbb{R}$ is necessary and $D^+(Z) > \mathbb{R}$ is sufficient for Z to be an A^{\otimes} zero set.

Note that for other spaces $A^{<k>}$ such that k has faster than logarithmic growth, a similar description of zero sets is not known.

4 Problem (B) for $A^{<k>}$

THEOREM 4.1

$$J(E; \mathbb{R}; k) \leq \max_{E \subset Z} \left[\int_E f^{\otimes}(\Pr E) + \log \int_E f^{\otimes}(\Pr E) \right] + (\log p) \frac{\mathbb{R}}{1 + \mathbb{R}} (1 + \int_E T(s) g) + \left(\frac{L}{\mathbb{R}}\right)^{\log_2 C} \text{Entr}_k(E)$$

where $0 < p < 1$, $0 < \mathbb{R} < \frac{1}{2}$ are arbitrary, C is the constant in (3), L is an absolute constant, and $\text{Entr}_k(E)$ is the k -entropy of E , defined as follows:

$$\text{Entr}_k(E) = \int_{\bigcup_{j=1}^n I_{n,j}} k(t) dt$$

where $I_{n,j}$ are the complementary arcs of E :

Special cases: (1) $E = \partial\mathbf{D}$: Letting $p \rightarrow 0^+$; we get

$$J(\partial\mathbf{D}; \rho; k) \cdot \int_{\partial\mathbf{D}} \log \rho^{-1} dm(z)$$

which is the classical Jensen's inequality (in fact, equality.)

(2) If $0 < \rho < 1$ on E and $p = \max_{z \in E} \rho^{-1}(z)$; we obtain

$$J(E; \rho; k) \cdot (\log p) \frac{jEj}{1} + \left(\frac{L}{jEj}\right)^{\log_2 C} \text{Entr}_k(E):$$

Choosing $\rho = jEj^{-2}$; we get

$$J(E; \rho; k) \cdot \frac{1}{2} (\log p) jEj + \left(\frac{2L}{jEj}\right)^{\log_2 C} \text{Entr}_k(E):$$

(3) If $p = 1$ and $\rho = \frac{1}{2}$; then

$$J(E; \rho; k) \cdot \int_E \log^+ \rho^{-1} dm(z) + (2L)^{\log_2 C} \text{Entr}_k(E):$$

Proof: Write

$$\partial\mathbf{D} \setminus E = \bigcup_n I_n$$

where the I_n are open disjoint arcs on the unit circle. Call a_n and b_n the endpoints of I_n : Let $0 < \rho < \frac{1}{2}$: Let ρ_n be the open arc of the circle inside the unit disk passing through a_n and b_n and forming an angle of $\frac{1}{4}\rho$ (we will think of it as the normalized angle ρ) with the arc I_n : Let $\tilde{E} = \bigcup_n \rho_n$: $\tilde{E} \cup E$ forms the boundary of an open subset \tilde{D} of the unit disk containing the origin. For the proof, we construct three functions U_1 ; U_2 ; and V as follows.

Step 1: Construction of U_1 and U_2 :

Define

$$U_1(z) = \int_E \text{Re} \left(\frac{3+z}{3-jz} \right) dm(z):$$

U_1 is the harmonic measure of E with respect to \mathbf{D} :

LEMMA 4.1

$$\lim_{r \rightarrow 1^-} U_1(r^3) = \hat{A}_E(z) \text{ a.e. on } \partial\mathbf{D}$$

where \hat{A}_E is the characteristic function of E : In addition, $U_1(z) \leq \rho$ for $z \in \tilde{D}$:

Assume $Entr_k(E)$ is finite and define the following harmonic function

$$V(z) = \int_{\partial\mathbf{D}} z \, d\mu$$

By relabeling L ; we get the statement of the lemma. \square

Step 3: Construction of H and application of the maximum principle. Finally, let us define

$$H(z) = U_2(z) + (\log p) \frac{1}{1 - U_1(z)} + L$$

$$\lim_{r \downarrow 1} f(r^3)$$